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Differential Equations for a Dynamic Thermal Conductivity Experiment[1]

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The mathematical model that describes a dynamic thermal conductivity experiment is reconsidered by taking into account the role of thermal expansion. Two differential equations are presented that take into account the various physical phenomena occurring in a long thin rod directly heated by a current pulse. One of the two equations keeps variables space and time completely separate and is particularly useful for computer simulations.

KEY WORDS: dynamic measurements; high temperature; scanning pyrometry; temperature profiles; thermal conductivity.

1. INTRODUCTION

A new technique for a dynamic experiment to measure thermal conductivity over a wide temperature range is currently in development at the Istituto di Metrologia "G. Colonnetti" (IMGC). The experimental apparatus and the measurement technique have been described elsewhere [1]; measurements on niobium are in progress. The experiment consists in bringing the specimen to high temperatures with a subsecond current pulse and in measuring during the free cooling (lasting 10–20 s) the evolving temperature profiles via high-speed scanning pyrometry [2].

In dynamic thermal conductivity experiments such as the one described before, thermal expansion plays an important role, on account of both its effects on the different thermophysical properties and its influence

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on the measurement technique. A microsecond-time resolution scanning pyrometer will always measure temperatures in predefined positions in "absolute space," but during the experiment it will measure the temperature of different points of the sample on account of the specimen's thermal expansion.

The different roles that thermal expansion plays in dynamic conductivity experiments have led us to reconsider its effects both on the mathematical model and on the different approaches that may be used to treat experimental data in these experiments. This paper considers two differential equations that describe the various physical phenomena taking place in a long thin rod directly heated by a current pulse. Both equations of linear heat flow take fully into account thermal expansion effects.

2. FIRST DIFFERENTIAL EQUATION

We consider a long thin rod rigidly constrained at the origin of the coordinate system (Fig. 1). The specimen is fixed at $x=0$ and thermal expansion takes place in the positive x-direction. The rod is heated by the electrical current I; at the end of the heating period the current is cut off and the specimen cools freely to room temperature. The task is to write the differential equation that describes the transient temperature evolution of each point of the rod taking into account all different physical phenomena taking place, including the thermal expansion of the rod. Of course the various thermophysical properties must be considered with their full temperature dependence, including that part deriving from thermal expansion effects [3]. The only approximation is the assumption of no radial temperature gradient in the specimen, leading to what is known as the "long thin rod approximation" [4].

The physical phenomena taking place in a dynamic thermal conductivity experiment are the same ones occurring in a steady-state experiment,

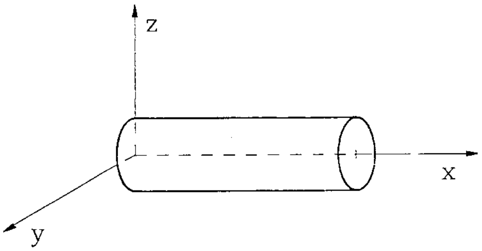

Fig. 1. Specimen and coordinate axes system. The rod is constrained at $x=0$ and expands in the positive x direction.

with the addition of terms related to the heat capacity and to the temperature dependence on time. The steady-state mathematical model is described in detail by Flynn [5]. A summary of steady-state experiments derived from this model is given by Taylor [4, 6].

The derivation of the differential equation is based on the power balance of the material contained in an infinitely small control volume fixed in space (Fig. 2). The term on the right-hand side of Eq. (1) is added to the steady-state model to refer to a dynamic experiment. The equation

$$Q_C + Q_R + Q_J + Q_T = Q_A \tag{1}$$

contains terms related to thermal conductivity, Q_C; radiation losses on the surface of the control volume, Q_R; heat absorbed (or released) in the control volume, Q_A; Joule heat, Q_J; and the Thomson heat, Q_T. Taking into account the functional dependence on temperature (both of thermophysical properties and of geometrical quantities), one obtains from Eq. (1)

$$\frac{\partial}{\partial x}\left(\lambda S \frac{\partial T}{\partial x}\right) - \varepsilon_{\mathrm{ht}} \sigma p (T^4 - T_a^4) + \frac{\rho I^2}{S} - \mu I \frac{\partial T}{\partial x} = \delta S c_{\mathrm{p}} \frac{DT}{Dt} \tag{2}$$

The quantities in Eq. (2) are as follows: λ, thermal conductivity; $\varepsilon_{\mathrm{ht}}$, hemispherical total emittance; ρ, electrical resistivity; μ, Thomson coefficient; c_{p}, heat capacity; δ, density; p, perimeter; S, cross-sectional area; σ, Stephan–Boltzmann constant; T, temperature; T_{a}, ambient temperature; and t, time. Temperature derivatives with respect to space $\partial T/\partial x$, $\partial^2 T/\partial x^2$, and time DT/Dt are also present in Eq. (2).

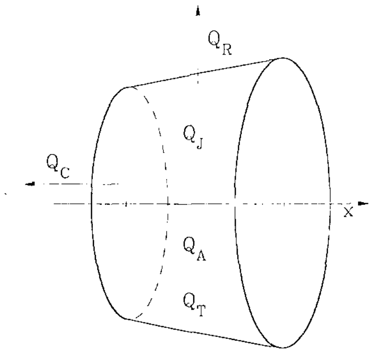

Fig. 2. Control volume used to derive the first differential equation. The volume is cone-shaped to indicate an exaggerated thermal expansion effect.

The power balance of the free cooling experiment (current $I=0$) is immediately obtained from Eq. (2) as

$$\frac{\partial}{\partial x}\left(\lambda S\frac{\partial T}{\partial x}\right)-\varepsilon_{\mathrm{ht}}\sigma p(T^4-T_{\mathrm{a}}^4)=\delta S c_{\mathrm{p}}\frac{DT}{Dt} \tag{3}$$

It is important to notice the form of the term Q_{A} [right-hand side of Eqs. (1)–(3)], which includes the time dependence and represents the main difference with respect to the steady-state model. Equations (1), (2), and (3) are expressions of the principle of conservation of energy. This principle applies to fixed quantities of matter that undergo changes of conditions. Therefore, extension of the steady-state case to a dynamic experiment implies that the time changes happening in the control volume must be computed by following in time the small amount of material that happened to be inside the control volume at the moment in which the equation is valid. A similar case is found in the motion of a fluid medium and $D(\ )/Dt$ denotes the "differentiation following the motion" [7]. Consequently the use of the "material" derivative [8] $D(\ )/Dt$ is necessary in dynamic thermal conductivity experiments. The concept is illustrated graphically in Fig. 3 in three dimensions, where some temperature profiles in a cooling

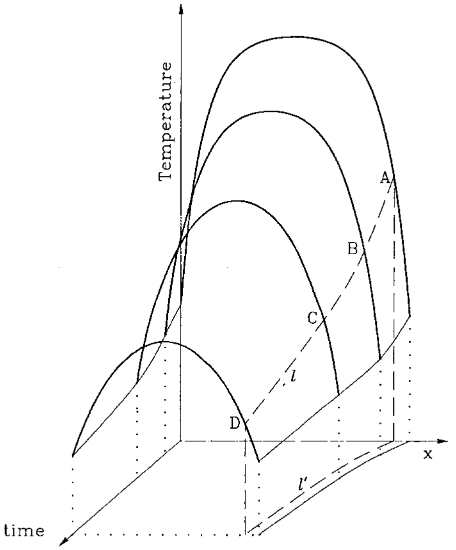

Fig. 3. Typical temperature profiles during a cooling experiment in a three-dimensional representation. The first differential equation is valid along line *l*.

experiment are plotted, showing their time evolution. Points A, B, C, and D (and all other possible points along line l) indicate the positions assumed by the same infinitesimal mass element of the specimen during cooling. Equation (3) may be applied by considering that the mass element moves due to the contraction of the specimen (cooling experiment). This motion must be taken into account, and when writing the differential equations for points A, B, C, and D, the derivative DT/Dt must be computed along the dashed line l. From Fig. 3 (that is not drawn proportionally to make the thermal expansion effect more evident), one notices that the projection of line l into the x, t (time) plane (dashed line l') is not perpendicular to the x axis. Mathematically this means that x and t are not completely separate but are somehow correlated through the thermal expansion of the material.

3. SECOND DIFFERENTIAL EQUATION

In certain cases (for example, in computer simulations) it might be useful to separate completely variables x and t. If temperature $T = T(x, t)$ and the two variables are uncorrelated, by the definition of the "material" derivative [8]

$$\frac{DT(x,t)}{Dt} = \frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} \tag{4}$$

where u_x is the speed of the portion of the moving specimen inside the control volume. In this case one must consider two processes happening in the control volume during an infinitesimal time. The first one is the change of heat stored in the control volume without considering that different portions of the specimen are inside the control volume at different times, and the second one is the change due to mass flow through the control volume (convection term).

Textbooks concerned with the heat conduction in solids rarely treat this case, because for most steady-state experiments (and for transient experiments related to thermal diffusivity) the properties are determined in small temperature ranges and thermal expansion plays a minor role. On the other hand, convection plays a major role in fluids and a complete derivation of this approach is given in the book by Landau and Lifshitz [9]. From the law of conservation of energy they derive a general heat transfer equation where the time-dependent term is expressed by

$$\delta c_{\mathrm{p}} \left(\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T \right) \tag{5}$$

In the dynamic thermal conductivity experiment the moving fluid is the expanding specimen and one must consider mass flow of the specimen through the control volume during the experiment. Of course, mass flow occurs in one direction during heating (expansion) and in the opposite direction during cooling (contraction).

Equation (4) may be substituted into Eq. (3) to yield

$$\frac{\partial}{\partial x}\left(\lambda S\frac{\partial T}{\partial x}\right)-\varepsilon_{\mathrm{ht}}\sigma p(T^4-T_{\mathrm{a}}^4)=\delta S c_{\mathrm{p}}\frac{\partial T}{\partial t}+\delta S c_{\mathrm{p}} u_x\frac{\partial T}{\partial x} \qquad (6)$$

The derivatives appearing in Eq. (6) are computed in the same position in "absolute space" (along the dashed line m in Fig. 4) and hence do not refer to the same mass element of the specimen (but refer to the portion of specimen that happened to be inside the control volume during the experiment). In other form, the projection of line m into the x, t plane (dashed line m' in Fig. 4) is perpendicular to the x axis. This means that all the points on line m have the same x coordinate (including points E, F, G, and H where line m crosses the temperature profiles).

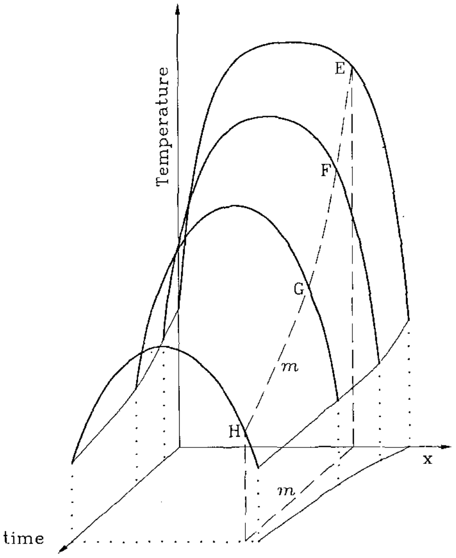

Fig. 4. Typical temperature profiles during a cooling experiment in a three-dimensional representation. The second differential equation is valid along line m.

The last term appearing in Eq. (6),

$$\delta S c_{\mathrm{p}} u_x \frac{\partial T}{\partial x} \tag{7}$$

represents the power contribution due to mass flow (convectional term). Since $u_x(x, t)$ is a quantity difficult to determine experimentally, it would be useful to find some conservation law to avoid measurement of this term. The useful relation is the mass conservation law, also known as the continuity equation [8]:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot (\delta \vec{u}) = 0 \tag{8}$$

where $\vec{u}$ is the vector of mass motion. This three-dimensional form of the continuity equation can be applied to the "long thin rod approximation" by defining a one-dimensional density,

$$w = \delta S \tag{9}$$

where w is the mass per unit length of the specimen. By analogy with Eq. (8), the one-dimensional continuity equation is

$$\frac{\partial w}{\partial t} + \frac{\partial (w u_x)}{\partial x} = 0 \tag{10}$$

Equation (10) may be integrated taking into account that, by definition, the rod is constrained at the origin of the coordinate system ($u_x = 0$ at $x = 0$). Consequently

$$w u_x = -\int_0^x \frac{dw}{dT} \frac{\partial T}{\partial t} \, dx \tag{11}$$

where the quantity dw/dT can be easily computed since it depends only on thermal expansion effects. Substituting Eq. (11) into Eq. (6), one obtains the final differential equation for the cooling experiment,

$$\frac{\partial}{\partial x}\left(\lambda S \frac{\partial T}{\partial x}\right) - \varepsilon_{\mathrm{ht}} \sigma p (T^4 - T_{\mathrm{a}}^4) = w c_{\mathrm{p}} \frac{\partial T}{\partial t} - c_{\mathrm{p}} \frac{\partial T}{\partial x} \int_0^x \frac{dw}{dT} \frac{\partial T}{\partial t} \, dx \tag{12}$$

where the variables x and t are completely separated.

The dynamic thermal conductivity experiment [1] is inherently more accurate for measurements during cooling and Eq. (12) was derived for this case. The derivation involved only the term on the right-hand side of

Eq. (2), so the same reasoning may be applied to a dynamic heating experiment, leading to

$$\frac{\partial}{\partial x}\left(\lambda S \frac{\partial T}{\partial x}\right) - \varepsilon_{\mathrm{ht}} \sigma p (T^4 - T_{\mathrm{a}}^4) + \frac{\rho I^2}{S} - \mu I \frac{\partial T}{\partial x}$$

$$= w c_{\mathrm{p}} \frac{\partial T}{\partial t} - c_{\mathrm{p}} \frac{\partial T}{\partial x} \int_0^x \frac{dw}{dT} \frac{\partial T}{\partial t}\, dx \qquad (13)$$

4. CONCLUSIONS

The thermal model applicable to dynamic thermal conductivity experiments has been reconsidered, leading to two one-dimensional partial differential equations. In both cases the "long thin rod approximation" is assumed and the derivation of the heat equation is based on the power balance of an infinitesimal mass of material contained in a control volume fixed in space. The main difference between the equations is that, in the first case, the temperature derivative with time is computed by following the mass element (material derivative). In the second case the partial time derivative is used, but one must consider mass flow through the control volume during the experiment. Both equations may be used to determine thermal conductivity from experimental temperature profiles, but the second equation might be particularly useful for computer simulations on account of the complete separation between variables space and time.

The work in this area is continuing with a complete computer simulation of the dynamic thermal conductivity experiment and with the search for different approaches in the treatment of experimental data.

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